Lecture 20.

Wednesday, November 20, 2019 5:36 AM

Thm 1. Let f be anal. in ~~region~~ connected region $G \subseteq \mathbb{C}$. TFAE:

(a) $f \equiv 0$.

(b) For some $a \in G$, $f^{(n)}(a) = 0$, $\forall n \in \mathbb{N}$.

(c) $Z_f := \{z \in G : f(z) = 0\}$ has a limit point in G. (Note: Z_f may well have limit pts in ∂G w/ consequence.)

Pf. Clearly (a) $\Rightarrow$ (b), (c). Going to show (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

(c) $\Rightarrow$ (b). Let a be limit point of Z_f, $\{a_n\}_{n=1}^{\infty}$ a seq. of distinct pts in Z_f $a_n \to a$. Since f cont. at $a \Rightarrow a \in Z_f$.
Suppose $f^{(m)}(a) \neq 0$ for some m, and let m be smallest integer s.t. this holds. Then, if $B(a,R) \subseteq G$, we have

$$f(z) = \sum_{n=m}^{\infty} a_n (z-a)^n, \quad a_m = \frac{f^{(m)}(a)}{m!} \neq 0.$$

$$\Rightarrow f(z) = (z-a)^m \underbrace{\sum_{n=0}^{\infty} a_{n+m}(z-a)^n}_{R.O.C \geq R \rightsquigarrow g(z) \text{ analytic}} = (z-a)^m g(z), \quad g(a) = a_m \neq 0.$$

It follows that $\exists \varepsilon > 0$ s.t. $g(z) \neq 0$ in $B(a,\varepsilon) \Rightarrow f(z) \neq 0$ in $B(a,\varepsilon)$.
This contradicts a is limit point of Z_f.

(b) $\Rightarrow$ (a). If $B(a,R) \subseteq G$, then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$, but $a_n = \frac{f^{(n)}(a)}{n!} = 0$, $\forall n \Rightarrow$ $f \equiv 0$ in $B(a,R)$. Consider $A := \{z \in G : f^{(n)}(z) = 0, \forall n\}$. Just showed $B(a,R) \subseteq A$, so $A \neq \emptyset$. We shall show A open + closed in $G \Rightarrow A = G$ by connectedness. A is clearly closed, since each $f^{(n)}$ is continuous on G, so if $a_n \in A$, $a_n \to a$, $\Rightarrow$ $0 = f^{(n)}(a_k) \to f^{(n)}(a) \Rightarrow a \in A$. We also already showed that if $a \in A$, $B(a,R) \subseteq G$, then $B(a,R) \subseteq A$, so A is open.

$\Rightarrow A = G$.

[4]

Cor. ① If f anal. in region G, $f \not\equiv 0$, and $f(a)=0$, then $\exists m \in \mathbb{N}$ s.t. $f^{(m)}(a) \neq 0$ and $f(z)=(z-a)^m g(z)$, where g is analytic and $g(z)\neq 0$ in $B(a,\varepsilon)$. **Def.** m is the multiplicity (or order) of the zero of f at $z=a$.

② If f, g anal. in region G and $\{z \in G : f(z)-g(z)=0\}$ has a limit point in G, then $f \equiv g$ in G.

"Local Liouville":
Max Modulus Thm. If f is analytic in region G and $\exists a \in G$ s.t. $|f(z)| \le |f(a)|$, $\forall z \in G$, then f is constant.

Index of a closed curve.

Prop1 Let $\gamma : [0,1] \to \mathbb{C}$ be closed p-w smooth curve, $a \notin \{\gamma\}$. Then

$$n(\gamma, a) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} \in \mathbb{Z}. \text{ (integer)}.$$

Def. $n(\gamma,a)$ is index or winding # of γ wrt a.

Pf. Note that $n(\gamma,a) = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)dt}{\gamma(t)-a}$. Set $g(s) = \int_0^s \frac{\gamma'(t)dt}{\gamma(t)-a}$, $s\in[0,1]$

$$\text{FTC} \Rightarrow g'(s) = \frac{\gamma'(s)}{\gamma(s)-a} \iff \underbrace{\gamma'(s) - g'(s)(\gamma(s)-a)}_{e^{g}\frac{d}{dt}(e^{-g}(\gamma-a))} = 0.$$

$\Rightarrow h(s) := e^{-g(s)}(\gamma(s)-a)$ is constant. But $h(0) = e^{-g(0)}(\gamma(0)-a) = \gamma(0)-a$ and $h(1) = e^{-g(1)}(\gamma(1)-a) \underset{\gamma(0)=\gamma(1)}{=} e^{-g(1)}(\gamma(0)-a)$. Since $\gamma(0)\neq a$,

[illegible]

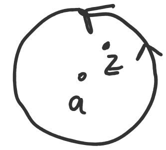
$$\underset{\gamma(0)=\gamma(1)}{}$$

$\Rightarrow e^{-g(1)} = 1 \Rightarrow g(1) = 2\pi i n$, some $n \in \mathbb{Z}$. Now,

$$n(\gamma, a) = \frac{1}{2\pi i} g(1) = n \in \mathbb{Z}. \quad \square$$

Ex: ① Previously we showed that w/ $\gamma = a + re^{2\pi i t}$, $t \in [0,1]$

$$n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z} = 1, \quad \forall z \in B(a, r)$$

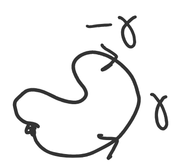


② Same argument shows that if $\gamma_k = a + re^{2\pi i k t}$ (traversing circle $|z-a| = r$ k times in $\operatorname{sgn}(k)$ direction), then

$$n(\gamma_k, z) = k, \quad \forall z \in B(a, r)$$

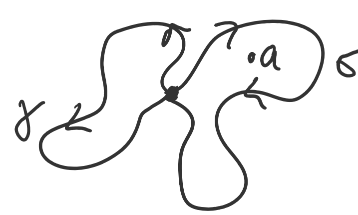
In general:

Basic Props. ① $n(-\gamma, a) = -n(\gamma, a)$



② If γ, σ closed w/ same start (end) pt, then

$$n(\gamma + \sigma, a) = n(\gamma, a) + n(\sigma, a).$$

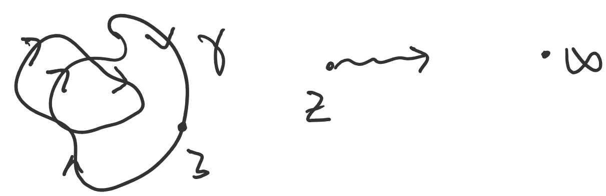


Rem. γ closed curve $\Rightarrow \{\gamma\}$ compact in $\mathbb{C} \Rightarrow \{\gamma\}$ compact in $\mathbb{C}_\infty$ and $\infty \notin \{\gamma\}$. $\Rightarrow \mathbb{C}_\infty \setminus \{\gamma\}$ is open w/ components $D_\infty, D_1, D_2, \ldots$ (in fact, only finitely many D_k) where $\infty \in D_\infty$ ← unbounded component many D_k) where $\infty \in D_\infty$ ← unbounded component

Thm 1. Let γ closed, p-w smooth curve in $\mathbb{C}$. Then, $n(\gamma, z)$ is constant on each component D_k, and $n(\gamma, z) = 0$ on D_∞.

Pf. By analytic Leibniz (HW problem), $n(\gamma, z)$ is analytic on each component D_k. ($\Rightarrow$ cont.) Since it only takes integral values, it must be constant on each D_k by intermediate value thm. Remains to show $n(\gamma, z) = 0$ on D_∞. We note, as $|z| \to \infty$,

$$|n(\gamma, z)| \le \frac{1}{2\pi} \int_\gamma \frac{|d\zeta|}{|\zeta - z|} \le \max_{\zeta \in \{\gamma\}} \frac{1}{|z| - |\zeta|} \cdot \frac{\text{length of } \gamma}{2\pi} \to 0.$$



Since $n(\gamma, z)$ constant on D_∞, it must be 0. ☒